WREATH PRODUCTS

VIV

1. SEMIDIRECT PRODUCTS

Wreath products are a very unusual way of making a big group out of smaller ones. The most common way to do this is called the *direct product* or *direct sum*.

Definition 1. The *direct product* of two groups K and H is the group $K \times H$ whose elements are the set

$$\{(k,h) \mid k \in K, \in H\}$$

and whose multiplication is given by

$$(k_1,h_1)(k_2,h_2) = (k_1k_2, h_1h_2).$$

Example 2. The Klein four group is the direct product $\mathbb{Z}/2 \times \mathbb{Z}/2$, and $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$.

But, our first step towards wreath products is first looking at *semi-direct products*, a generalization of direct products.

Definition 3. Let K and H be two groups with an action of K on H, i.e. a homomorphism

$$\alpha : K \to \mathrm{Aut}(H).$$

Then the *semidirect product* $K \ltimes H$ is the group whose elements are

$$\{(k,h) \mid k \in K, \in H\}$$

and whose multiplication is given by

$$(k_1,h_1)(k_2,h_2) = (k_1k_2, h_1 \cdot \alpha(k_1)(h_2)).$$

That's quite a strange multipliction law, so let's understand it by looking at the example of dihedral groups.

Example 4. Consider the dihedral group D_n, given by the presentation

$$D_n = \langle r, f \mid r^n = f^2 = 1, rf = fr^{-1} \rangle.$$

We'd like to think of it as a product of the rotation subgroup and the flip subgroup, but it's not quite a direct product. So we need a semidirect product, which means we need an appropriate map

$$\alpha : \mathbb{Z}/2 \to \mathrm{Aut}(\mathbb{Z}/n).$$

In the end, this map is given by $\alpha(\mathrm{id}) = \mathrm{id}$ and $\alpha(f) = (r \mapsto r^{-1})$.

Then $\mathbb{Z}/2 \ltimes_\alpha \mathbb{Z}/n = \{(1,1), \ldots, (1,r^{n-1}), (f,1), (f,r), \ldots, (f,r^{n-1})\}$, such that

$$(f,1)(1,r)(f,1) = (f,1)(f,r) = (1, \alpha(f)(r)) = (1, r^{-1}).$$

Exercise 5. Prove that $\mathbb{Z}/2 \ltimes_\alpha \mathbb{Z}/n \cong D_n$ via the map $\varphi : \mathbb{Z}/2 \ltimes_\alpha \mathbb{Z}/n \to D_n$ given by $\varphi(f^a, r^b) = r^b f^a$.

Exercise 6. For a semidirect product $K \ltimes_\alpha H$, show that H is a normal subgroup of $K \ltimes_\alpha H$.

2. Semidirect Products and Group Presentations

Let's look at the group presentation of a semidirect product. Let $K = \langle X \mid R \rangle$, where X is a set of generators and R is a set of relations, and similarly let $H = \langle Y \mid S \rangle$, with $\alpha : K \to \mathrm{Aut}(H)$ a homomorphism. Then the group presentation for $G \ltimes_\alpha H$ is given by

$$\left\langle X \cup Y \mid R \cup S \cup \{xyx^{-1} = \alpha(x)(y) \mid x \in X, y \in Y\} \right\rangle.$$

Exercise 7. Show that this group presentation is isomorphic to $G \ltimes_\alpha H$ as before, by constructing an isomorphism from the presentation to our previous definition of $G \ltimes_\alpha H$.

3. Internal Semidirect Products

Sometimes, rather than starting with two groups H and K, we'd like to decompose a group G as a semidirect product of two of its subgroups.

Proposition 8. *Let G be a group with H and K subgroups such that*

(1) $H \trianglelefteq G$
(2) $H \cap K = \{1\}$
(3) $HK = G$.

Then $G \cong K \ltimes H$, wehre $\alpha : H \to \mathrm{Aut}(K)$ is given by $\alpha(h)(k) = hkh^{-1}$.

Proof. Consider the map $\varphi : K \ltimes H \to G$ via $(k, h) \mapsto hk$. This map is an isomorphism, i.e. it is a homomorphism that is injective and bijective. □

This means that most of the time when we'd *like* to split G into a product of two subgroups, it is a semidirect product of those two subgroups.

4. The Lamplighter Group

Let's talk about one of my favorite groups, the lamplighter group. The setting is as follows. let's say you have an infinite street with lamps at every integer point, and a lamplighter whose job is to light the lamps. The lamplighter starts at a point 0, and sets of instructions are of the form "flip this set of lamps, then walk to that endpoint." Two elements compose via doing successive instructions. In other words, the composition of two instructions A and B is the instructions to do A, then do B as if your end location from A is your new start point.

OK, but that's all well and confusing. In practice, we can think of elements as the set of pairs $(m, p(x))$, with $p(x) \in \mathbb{Z}/2[x, \frac{1}{x}]$ a polynomial dictating which lights to flip, and $m \in \mathbb{Z}$ is the integer value of the lamplighter's final location. Then multiplication is given by

$$(m_1, p_1(x))(m_2, p_2(x)) = (m_1 + m_2, p_1(x) + x^{m_1} p_2(x)).$$

But, this looks like a semidirect product! Let H be the direct sum indexed by $\mathbb{Z}$ of copies of $\mathbb{Z}/2$, i.e.

$$H = \bigoplus_{i \in \mathbb{Z}} (\mathbb{Z}/2) = \{\text{polynomials in } \mathbb{Z}/2[x, \tfrac{1}{x}] \text{ under addition}\}.$$

Meanwhile, let $K = \mathbb{Z}$. Then we have a map $\alpha : K \to \mathrm{Aut}(H)$ given by, for $m \in K = \mathbb{Z}$,

$$\alpha(m)(p(x)) = x^m p(x).$$

Note that we can consider K and H to be subgroups of the lamplighter group G, via $K = \{(m, p(x)) \mid p(x) = 0\}$ and $H = \{(m, p(x)) \mid m = 0\}$.

Exercise 9. With H, K subgroups of G as above, show that H is a normal subgroup of G, and that for $m \in K$ and $p(x) \in H$, $\alpha(m)(p(x)) = (m, 0)(0, p(x))(m, 0)^{-1}$.

For our H and K, we also know that $HK = G$ and $H \cap K = \{(0,0)\} = \{\mathrm{id}\}$. This implies that $G = K \ltimes H = \mathbb{Z} \ltimes \bigoplus_{i \in \mathbb{Z}} (\mathbb{Z}/2)$.

But somehow it's a little dissatisfying to have this as just a semidirect product. In particular, the two copies of $\mathbb{Z}$ in the above expression are truly the same $\mathbb{Z}$. It's a wreath product!

Definition 10. Let H and K be groups. Let A be the direct sum given by

$$A = \bigoplus_{w \in K} H_w,$$

where each H_w is an identical copy of H, indexed by elements of K. Then K acts on A via

$$k(a_w) = a_{k^{-1}w}.$$

The *restricted wreath product* $H \wr K$ is the group $H \wr K = K \ltimes A$.

Exercise 11. Prove that the above action of K on A describes a homomorphism $K \to \mathrm{Aut}(A)$.

Then the lamplighter group can be expressed as $\mathbb{Z}/2 \wr \mathbb{Z}$, which truly seems much cleaner.

Exercise 12. Assume that K and H are both finite. What is the order of $H \wr K$ in terms of the order of H and the order of K?